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Cambridge Pre-U Certificate

FURTHER MATHEMATICS **1348/01**

Paper 1 Further Pure Mathematics **May/June 2017**

MARK SCHEME

Maximum Mark: 120

Published

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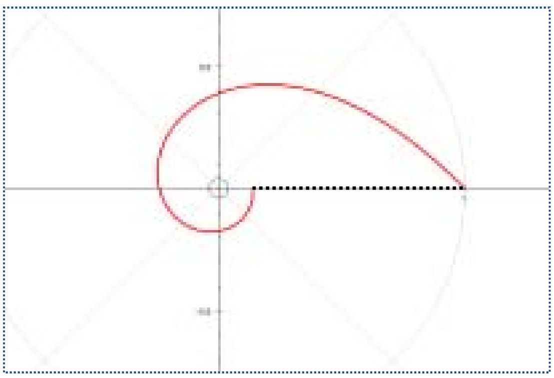
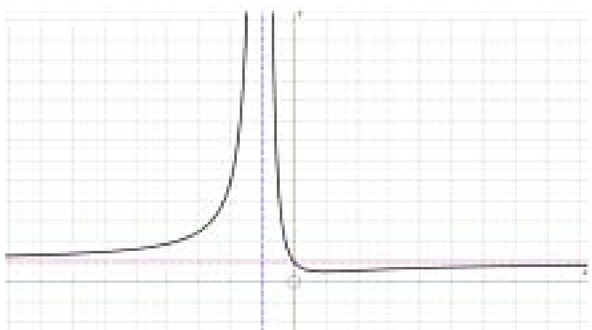
This document consists of **11** printed pages.

Question	Answer	Marks	Part Marks
1	$(a + \mathrm{i}b)^2 = (a^2 - b^2) + \mathrm{i}.2ab$	B1	
	$(a^2 - b^2) = 21$ and $ab = -10$	M1	Comparing real and imaginary parts
	e.g. eliminating one variable and solving for the other	M1	Allow implied by e.g. $a = 5$, $b = 2$ (or v.v.)
	$a = \pm 5,\ b = \mp 2$	A1	Ignore any complex answers
2	$\Sigma\alpha = -2$ and $\Sigma\alpha\beta = 3$	B1	Both ($\alpha\beta\gamma = -7$ not required)
	$\alpha^2 + \beta^2 + \gamma^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta = -2$	M1A1	FT
	1 real and 2 complex (conjugate) roots	B1	Accept any comment that "not all roots are real
	Alternative Form an equation with roots $\alpha^2, \beta^2, \gamma^2$; $y^3 + 2y^2 - 19y - 49 = 0$	M1A1	
	$\Sigma\alpha^2 = -\dfrac{b}{a} = -2$	B1	FT
	1 real and 2 complex (conjugate) roots	B1	Accept any comment that "not all roots are real
3(i)		B3	**B1** Starts at (1, 0) **B1** Decreasing spiral **B1** All (essentially) correct
3(ii)	Area $= \frac{1}{2}\displaystyle\int_0^{2\pi} \frac{1}{(1+\theta)^2}\,\mathrm{d}\theta$	M1	Attempt to integrate $k(1 + \theta)^{-2}$
	$= \frac{1}{2}\left[\dfrac{-1}{1+\theta}\right]_0^{2\pi}$	A1	Correct integration
	$= \frac{1}{2}\left(1 - \dfrac{1}{1+2\pi}\right)$ or $\dfrac{\pi}{1+2\pi}$	A1	Correct answer

Question	Answer	Marks	Part Marks
4	$\dot{x} = t - \frac{1}{t}$ and $\dot{y} = 2$	**B1**	at least $\dot{x}$ correct
	$(\dot{x})^2 + (\dot{y})^2 = t^2 - 2 + \frac{1}{t^2} + 4$	**M1**	attempted
	$= \left(t + \frac{1}{t}\right)^2$	**A1**	Here or in the integral for S (2nd fraction of line below)
	$S = 2\pi\int_1^4 2t\,.\left(t + \frac{1}{t}\right)\mathrm{d}t$	**M1**	Use of formula (Ignore limits until final answer)
	$= 4\pi\int_1^4 \left(t^2 + 1\right)\mathrm{d}t$	**A1**	In a form ready to integrate
	$= 4\pi\left[\frac{t^3}{3} + t\right]_1^4$	**B1**	Correct integration (**FT** provided it is polynomial)
	$= 96\pi$	**A1**	
5(i)	$y = \tanh^{-1} x \Leftrightarrow \tanh y = x = \frac{\mathrm{e}^{2y} - 1}{\mathrm{e}^{2y} + 1}$	**M1**	
	$x\mathrm{e}^{2y} + x = \mathrm{e}^{2y} - 1 \Leftrightarrow 1 + x = \mathrm{e}^{2y}(1 - x)$	**M1**	Identifying e^{2y}
	$y = \tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$	**A1**	Legitimately obtained by taking logs Allow verification by substitution of given result
5(ii)	**Method I** $t + \frac{1}{t} = 4 \Rightarrow t^2 - 4t + 1 = 0$	**M1**	Creating a quadratic in tanh x
	$\Rightarrow t = 2 \pm \sqrt{3}$	**M1**	Solving
	Using $\frac{1}{2}\ln\left(\frac{1+t}{1-t}\right)$ with $t = 2 - \sqrt{3}$ and/or $2 + \sqrt{3}$	**M1**	(NB since $\lvert \tanh x \rvert < 1$, it must be $t = 2 - \sqrt{3}$)
	$x = \frac{1}{2}\ln\left(\frac{3-\sqrt{3}}{-1+\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}}\right) = \frac{1}{2}\ln\left(\sqrt{3}\right)$	**M1**	By rationalising denominator or direct observation (possibly from calculator use)
	$= \frac{1}{4}\ln(3)$	**A1**	Must be in this form

Question	Answer	Marks	Part Marks
5(ii)	**Method II** $\frac{\text{sh}}{\text{ch}} + \frac{\text{ch}}{\text{sh}} = 4$	**M1**	
	$\Rightarrow \text{ch}^2 + \text{sh}^2 = 4\text{sh.ch} \Rightarrow \cosh(2x) = 2\sinh(2x)$	**M1**	Conversion to double-"angles"
	$\Rightarrow \tanh(2x) = \frac{1}{2}$	**A1**	
	$\Rightarrow 2x = \frac{1}{2}\ln\left(\frac{\frac{3}{2}}{\frac{1}{2}}\right)$	**M1**	Use of $\tanh^{-1}x$ formula from **(i)**
	$\Rightarrow x = \frac{1}{4}\ln(3)$	**A1**	Must be in this form
	Method III $\frac{e^{2x}-1}{e^{2x}+1} + \frac{e^{2x}+1}{e^{2x}-1} = 4$	**M1**	
	$\Rightarrow (e^{2x}-1)^2 + (e^{2x}+1)^2 = 4(e^{2x}-1)(e^{2x}+1)$	**M1**	
	$\Rightarrow e^{4x} - 2e^{2x} + 1 + e^{4x} + 2e^{2x} + 1 = 4(e^{4x}-1)$	**A2**	**A1** LHS **A1** RHS
	$\Rightarrow 6 = 2e^{4x} \Rightarrow x = \frac{1}{4}\ln(3)$	**A1**	Must be in this form
6(i)	HA $y = 1$ VA $x = -1$	**B2**	**B1** for each
6(ii)	$y = \frac{x^2+1}{(x+1)^2}$ or $y = 1 - \frac{2x}{(x+1)^2}$ $\Rightarrow \frac{dy}{dx} = \frac{(x+1)^2(2x) - (x^2+1).2(x+1)}{(x+1)^4}$ or $-\frac{(x+1)^2.2 - 2x.2(x+1)}{(x+1)^4} = \frac{2(x-1)}{(x+1)^3}$	**M1A1**	Attempted; correct unsimplified
	$\Rightarrow \frac{dy}{dx} = 0$ when $x = 1,\ y = \frac{1}{2}$	**A2**	**A1** for each
6(iii)		**3**	**G1** for graph in 2 bits, separated by a (**FT**) vertical asymptote and all positive **G1** for y-intercept at (0, 1) and MIN. in (approx. **FT**) correct place **G1** for correct asymptotic behaviour

Question	Answer	Marks	Part Marks
7(i)	$y = k\,x\sin 2x \Rightarrow \dfrac{dy}{dx} = 2k\,x\cos 2x + k\sin 2x$	**M1**	attempt using the *Product Rule*
	and $\dfrac{d^2y}{dx^2} = k\,x\,.-4\sin 2x + 2k\cos 2x + 2k\cos 2x$	**M1**	attempt using the *Product Rule*
	$= -4y + 4k\cos 2x$	**M1**	for substn. into given d.e. or comparison
	$\Rightarrow k = 2$	**A1**	
7(ii)	Comp. Fn. from $m^2 + 4 = 0$	**M1**	
	$\Rightarrow y_C = A\cos 2x + B\sin 2x$	**A1**	Or $R\cos(2x - \alpha)$ etc.
	Gen. Soln. is thus $y = A\cos 2x + (B + 2x)\sin 2x$	**B1**	**FT**
	Then $\dfrac{dy}{dx} = -2A\sin 2x + 2(B + 2x)\cos 2x + 2\sin 2x$ OR $= 2(B + 2x)\cos 2x$ if found after A (correctly) evaluated	**B1**	
	Subst^g. in given initial conditions	**M1**	
	$A = 1$ from $x = 0, y = 1$	**A1**	**FT** from an incorrect $x\sin 2x$ term in y
	$B = \frac{1}{2}$ from $x = 0$, $\dfrac{dy}{dx} = 1$ i.e. soln. is $y = \cos 2x + (2x + \frac{1}{2})\sin 2x$	**A1**	**FT** from an incorrect $x\cos 2x$ term in y' Withhold final A mark if in e^complex form
8(i)(a)	$\cos\theta = \dfrac{12 + 2 + 6}{3 \times 7} = \dfrac{20}{21}$	**M1A2**	**A1** scalar product; **A1** both moduli Give B1s for correct scalar product; both moduli if $\sin\theta = \ldots$ used
8(i)(b)	Subst^g. $(2\lambda, -\lambda, 2\lambda)$ into $6x - 2y + 3z = 35$	**M1**	
	$\Rightarrow \lambda = \frac{7}{4} \Rightarrow \mathbf{p} = \frac{7}{4}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$	**A1A1**	Second **A1** is **FT**
8(i)(c)	SD O to $\Pi_1 = OP\cos\theta = \dfrac{7}{4} \times 3 \times \dfrac{20}{21} = 5$	**M1A1**	**A1FT**

Question	Answer	Marks	Part Marks
8(i)(c)	**Alternative I** $(6\lambda, -2\lambda, 3\lambda)$ in plane $\Rightarrow$ $36\lambda + 4\lambda + 9\lambda = 35$	**M1**	$\Rightarrow$ $\lambda = \frac{5}{7}$
	$\Rightarrow$ SD = $\lambda\sqrt{6^2+2^2+3^2} = 5$ cao	**A1**	
	Alternative II Quote formula: SD = $\left\lvert \frac{d}{\lvert \mathbf{n} \rvert} \right\rvert = \frac{35}{\sqrt{6^2+2^2+3^2}} = 5$ cao	**M1A1**	
8(ii)	Similar working gives $\lambda_1 = -\frac{21}{40}$	**B1**	
	Planes parallel, *and on opposite sides of O*, so total distance is $3\left(\frac{7}{4}+\frac{21}{40}\right)\cos\theta = \frac{13}{2}$	**M1A1**	
	Alternative I Π_2 has equation $\mathbf{r} \bullet \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix} = -\frac{21}{2}$	**B1**	
	$\Rightarrow$ SD to Π_2 is $-\frac{3}{2}$	**B1**	
	Planes parallel, *and on opposite sides of O*, so distance between them is $5 - -\frac{3}{2} = \frac{13}{2}$	**B1**	**FT**
	Alternative II Quote Sh. Dist. formula for $P\left(\frac{7}{4}, -\frac{7}{2}, \frac{7}{2}\right)$ to Π_2	**M1**	or using distance from any point in Π_1 or Π_2 to other plane
	SD = $\left\lvert \frac{12(\frac{7}{4}) - 4(-\frac{7}{2}) + 6(\frac{7}{4}) + 21}{\sqrt{12^2+4^2+6^2}} \right\rvert = \frac{91}{14} = \frac{13}{2}$	**A1A1**	
9(i)	Full elimination of x: $I = \int \frac{1}{\cosh^2\theta.\sinh\theta}.\sinh\theta \,\mathrm{d}\theta$	**M1**	
	$\Rightarrow$ $I = \int \operatorname{sech}^2\theta \; \mathrm{d}\theta$	**A1**	
	$= \tanh\theta$ $(+ C)$	**A1**	
	$= \frac{\sqrt{x^2-1}}{x}$ $(+ C)$ from $\frac{\sinh\theta}{\cosh\theta}$	**A1**	**(AG)**

Question	Answer	Marks	Part Marks
9(ii)	$\sec y = x \Rightarrow \sec y \tan y \dfrac{dy}{dx} = 1$	**M1A1**	
	Use of $\tan y = \sqrt{\sec^2 y - 1}$	**M1**	
	to get $\dfrac{dy}{dx} = \dfrac{1}{x\sqrt{x^2-1}}$	**A1**	**AG** Ignore lack of reason for taking the +ve sq.rt. (e.g. from +ve gradient of $\sec^{-1}$ curve)
9(iii)	$\int \sec^{-1} x \cdot \dfrac{1}{x^2}\, dx$ $= \sec^{-1} x \cdot \dfrac{-1}{x} - \int \dfrac{-1}{x} \cdot \dfrac{1}{x\sqrt{x^2-1}}\, dx$ $= \dfrac{-\sec^{-1} x}{x} + \int \dfrac{1}{x^2\sqrt{x^2-1}}\, dx$	**M1A2**	By parts
	$= \dfrac{-\sec^{-1} x}{x} + \dfrac{\sqrt{x^2-1}}{x}$ $(+ C)$	**A1**	using **(i)**
	Alternative Use $u = \sec^{-1} x \Rightarrow \dfrac{du}{dx} = \dfrac{1}{x\sqrt{x^2-1}}$ $\Rightarrow \sec u \tan u\, du = dx$	**M1**	
	$\Rightarrow \int \sec^{-1} x \cdot \dfrac{1}{x^2}\, dx = \int u \sin u\, du$	**A1**	
	2-stage integration by parts: $\int u \sin u\, du = -u \cos u + \int \cos u\, du$ $= -u \cos u + \sin u\ (+C)$	**M1**	
	Correctly turning this back into $= \dfrac{-\sec^{-1} x}{x} + \dfrac{\sqrt{x^2-1}}{x}$ $(+ C)$	**A1**	
10(i)	$\dfrac{1}{(k-1)k(k+1)} \equiv \dfrac{A}{k-1} + \dfrac{B}{k} + \dfrac{C}{k+1}$	**M1**	Correct form
	Equating terms / substn. / cover-up	**M1**	Method for determining constants
	$\equiv \dfrac{\frac{1}{2}}{k-1} - \dfrac{1}{k} + \dfrac{\frac{1}{2}}{k+1}$	**A1**	

Question	Answer	Marks	Part Marks
10(ii)	$\sum_{k=3}^{n} \frac{1}{(k-1)k(k+1)} \equiv \frac{1}{2}\sum_{k=3}^{n} \frac{1}{k-1} + \frac{1}{2}\sum_{k=3}^{n} \frac{1}{k+1} - \sum_{k=3}^{n} \frac{1}{k}$	M1	Splitting up
	$\equiv \frac{1}{2}\left\{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n-1}\right\} + \frac{1}{2}\left\{\frac{1}{4}+\ldots+\frac{1}{n-1}+\frac{1}{n}+\frac{1}{n+1}\right\} - \left\{\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n-1}+\frac{1}{n}\right\}$	M1	Attempt at cancelling of terms
	$\equiv \frac{1}{2}\left\{\frac{1}{2}+\frac{1}{3}\right\} + \frac{1}{2}\left\{\frac{1}{n}+\frac{1}{n+1}\right\} - \left\{\frac{1}{3}+\frac{1}{n}\right\}$	A1	Correct ones clearly identifed
	$\equiv \frac{1}{12} - \frac{1}{2}\left\{\frac{1}{n}-\frac{1}{n+1}\right\} \equiv \frac{1}{12} - \frac{1}{2n(n+1)}$	A1	Legitimately shown **(AG)**
	Limit (S_n) as $n \to \infty$ is $S = \frac{1}{12}$	B1	**FT**
	Alternative $\sum_{k=3}^{n} \frac{1}{(k-1)k(k+1)} \equiv \frac{1}{2}\sum_{k=3}^{n} \frac{1}{k(k-1)} - \frac{1}{2}\sum_{k=3}^{n} \frac{1}{k(k+1)}$	M1	
	$= \frac{1}{2}\left(\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\ldots+\frac{1}{n(n-1)}\right) - \frac{1}{2}\left(\frac{1}{12}+\frac{1}{20}+\ldots+\frac{1}{n(n-1)}+\frac{1}{n(n+1)}\right)$	M1	Clear listing of terms
	All correct and ready to cancel	A1	
	$= \frac{1}{12} - \frac{1}{2n(n+1)}$	A1	Legitimately shown **(AG)**
	Limit (S_n) as $n \to \infty$ is $S = \frac{1}{12}$	B1	**FT**
10(iii)	$k^3 > k^3 - k = k(k-1)(k+1)$ $\Rightarrow \frac{1}{k^3} < \frac{1}{(k-1)k(k+1)}$	B1	
10(iv)	$\sum_{k=1}^{\infty} \frac{1}{k^3} > 1 + \frac{1}{8} = \frac{9}{8} = \frac{27}{24}$	B1	Given result justified
	$\sum_{k=1}^{\infty} \frac{1}{k^3} = 1 + \frac{1}{8} + \sum_{k=3}^{\infty} \frac{1}{k^3} < 1 + \frac{1}{8} + \sum_{k=3}^{n} \frac{1}{(k-1)k(k+1)}$	M1	
	$= 1 + \frac{1}{8} + \frac{1}{12} = \frac{29}{24}$	A1	Given result justified
11(i)(a)	$\mathbf{AB} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$	B1	
	det $\mathbf{A} = ad - bc$ and det $\mathbf{B} = eh - fg$	B1	

Question	Answer	Marks	Part Marks
11(i)(b)	det(**AB**) = $(ae+bg)(cf+dh)-(af+bh)(ce+dg)$ and some attempt to multiply out	**M1**	
	$= acef + adeh + bcfg + bdgh$ $- acef - bceh - adfg - bdgh$ $= adeh - bceh - adfg + bcfg$ $= (ad-bc)(eh-fg)$	**A1**	Legitimately shown
11(ii)	*CLOSURE*: $\mathbf{A}, \mathbf{B} \in S \Rightarrow \det \mathbf{A} = \det \mathbf{B} = 1$	**M1**	Attempted
	and above result $\Rightarrow \det \mathbf{AB} = 1 \Rightarrow \mathbf{AB} \in S$ (*ASSOCIATIVITY*: given)	**A1**	Convincing
	IDENTITY: $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in S$ since det **I** = 1.1 – 0.0 = 1	**B1**	Must show why $\mathbf{I} \in S$ and not just say that **I** is the identity
	INVERSES: $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S \Rightarrow \mathbf{A}^{-1}$ $= \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in S$	**B1**	for stating $\mathbf{A}^{-1}$ (or explaining that it exists)
	Since $da - (-b)(-c) = ad - bc = 1$ Hence $(S, \times_{\mathbf{M}})$ is a group, G.	**B1**	for justifying its membership of S
11(iii)(a)	$\det \mathbf{K} = 1.0 - \mathrm{i}.\mathrm{i} = -\mathrm{i}^2 = 1$ (so $\mathbf{K} \in S$)	**B1**	
11(iii)(b)	Attempt at powers of **K**; $\mathbf{K}^2$ & $\mathbf{K}^3$	**M1**	
	$\mathbf{K}^2 = \begin{pmatrix} 0 & i \\ i & -1 \end{pmatrix}$ and $\mathbf{K}^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	**A1**	
	NB $\mathbf{K}^4 = \begin{pmatrix} -1 & -i \\ -i & 0 \end{pmatrix}$ and $\mathbf{K}^5 = \begin{pmatrix} 0 & -i \\ -i & 1 \end{pmatrix}$ $\Rightarrow \mathbf{K}^6 = \mathbf{I}$ and H has order $n = 6$	**A1**	
11(iii)(c)	e.g. The set of rotations about O through multiples of $60°$ OR ($\mathbf{K}$*) = group generated by $\begin{pmatrix} 1 & -i \\ -i & 0 \end{pmatrix}$	**B1**	**FT** for any n
	Justifying the two are isomorphic	**B1**	e.g. stating both are cyclic, etc.

Question	Answer	Marks	Part Marks
12(i)	**Method I** $\mathrm{F}_{n+2}(\theta) - \frac{1}{4}\sin^2(2\theta)\ \mathrm{F}_{n+1}(\theta)$ $\equiv (c^2+s^2)(c^{2n+4}+s^{2n+4})$ $\quad - \frac{1}{4}(2sc)^2(c^{2n+2}+s^{2n+2})$	**M2**	**M1** all F_n terms **M1** $\sin 2\theta$ form
	$\equiv c^{2n+6}+c^2s^{2n+4}+s^2c^{2n+4}+s^{2n+6}$ $\quad -c^2s^2(c^{2n+2}+s^{2n+2})$	**A1**	
	$\equiv c^{2n+6}+s^{2n+6} \equiv \mathrm{F}_{n+3}(\theta)$	**A1**	**AG**
	Method II $\equiv c^{2n+4}+s^{2n+4} - s^2c^2(c^{2n+2}+s^{2n+2})$	**M1**	Use of $\sin 2\theta$ form
	$\equiv c^{2n+4}+s^{2n+4} - s^2c^{2n+4} - c^2s^{2n+4}$	**A1**	
	$\equiv (1-s^2)c^{2n+4}+(1-c^2)s^{2n+4}$	**M1**	
	$\equiv c^{2n+6}+s^{2n+6} \equiv \mathrm{F}_{n+3}(\theta)$	**A1**	**AG**
12(ii)(a)	Use of $z = c + \mathrm{i}s$ and $z^{-1} = c - \mathrm{i}s$	**M1**	
	$z + z^{-1} = 2c$ and $z - z^{-1} = 2\mathrm{i}s$	**A2**	**A1** for each
12(ii)(b)	**Method I** $(2c)^6 = (z+z^{-1})^6 = z^6+6z^4+15z^2+20$ $\quad +15z^{-2}+6z^{-4}+z^{-6}$	**M1**	
	$= 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$	**A1**	
	$-(2s)^6 = (z-z^{-1})^6 = z^6-6z^4+15z^2-20$ $\quad +15z^{-2}-6z^{-4}+z^{-6}$ $= 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$	**B1**	**FT** (Must have – sign)
	Subtracting: $64(c^6+s^6) = 12(z^4+z^{-4}) + 40$ $= 12 \,.\, 2\cos 4\theta + 40$	**M1**	
	Dividing by 8: $8(c^6+s^6) = 3\cos 4\theta + 5$	**A1**	**AG**
	Use of $\cos 4\theta = 2\cos^2 2\theta - 1$ and $1 = \cos^2 2\theta + \sin^2 2\theta$	**M1**	
	$\Rightarrow c^6+s^6 = \frac{3}{8}(2\cos^2 2\theta) + \left(-\frac{3}{8}+\frac{5}{8}\right)(\cos^2 2\theta + \sin^2 2\theta)$ $= \cos^2 2\theta + \frac{1}{4}\sin^2 2\theta$	**A1**	**AG**

Question	Answer	Marks	Part Marks
12(ii)(b)	**Method II** $\cos 4\theta = \text{Re}(c + \text{i}s)^4$	**M1**	
	$= c^4 - 6c^2s^2 + s^4 = c^4 - 6c^2(1-c^2) + (1-c^2)^2$ $= 8c^4 - 8c^2 + 1$	**A1**	
	$c^6 + s^6 = c^6 + (1-c^2)^3 = c^6 + 1 - 3c^2 + 3c^4 - c^6$	**M1**	
	$= 3c^4 - 3c^2 + 1$	**A1**	
	so that $8(c^6 + s^6) = 3\cos 4\theta + 5$	**A1**	**AG**
	Use of $\cos 4\theta = \cos^2 2\theta - \sin^2 2\theta$ and $1 = \cos^2 2\theta + \sin^2 2\theta$	**M1**	
	$\Rightarrow 8(c^6 + s^6) = 3\cos 4\theta + 5$ $= 3(\cos^2 2\theta - \sin^2 2\theta) + 5(\cos^2 2\theta + \sin^2 2\theta)$ $\Rightarrow c^6 + s^6 = \cos^2 2\theta + \frac{1}{4}\sin^2 2\theta$	**A1**	**AG**
12(iii)	Case for $n = 1$ established in (ii) (b):	**B1**	noted explicitly (possibly at end)
	Assume $c^{2k+4} + s^{2k+4} \leqslant \cos^2 2\theta + \frac{1}{2^{k+1}}\sin^2 2\theta$	**B1**	i.e. the case for $n = k$
	A clear statement of the result must be given, possibly within what follows Then $c^{2k+6} + s^{2k+6} =$ $c^{2k+4} + s^{2k+4} - \frac{1}{4}\sin^2 2\theta(c^{2k+2} + s^{2k+2})$	**M1**	attempt at $n = k + 1$ case using **(i)**'s identity
	$\leqslant \cos^2 2\theta + \frac{1}{2^{k+1}}\sin^2 2\theta - \frac{1}{4}\sin^2 2\theta(c^{2k+2} + s^{2k+2})$	**M1**	use of the induction hypothesis (i.e. the $n = k$ case)
	$= \cos^2 2\theta + \frac{1}{2^{k+2}}\sin^2 2\theta - \frac{1}{4}\sin^2 2\theta\left(c^{2k+2} + s^{2k+2} - \frac{1}{2^k}\right)$	**M1A1**	splitting up the $\sin^2 2\theta$ term into two equal parts
	$\leqslant \cos^2 2\theta + \frac{1}{2^{k+2}}\sin^2 2\theta$ Proof follows by induction since $\sin^2 2\theta \geqslant 0$ and given result that $c^{2k+2} + s^{2k+2} \geqslant \frac{1}{2^k}$	**A1**	